University of Arkansas at little Rock

Department of Systems Engineering

SYEN 3314 Probability and Random Signals - Summer 2009

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Midterm 1 Solution

- This is a closed book exam.
- Calculators are not allowed.
- There are 8 problems on the exam plus one extra credit (or bonus) problem.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- A correct answer does not guarantee full credit, and a wrong answer does not guarantee loss of credit. You should clearly but concisely indicate your reasoning and show all relevant work. Your grade on each problem will be based on our assessment of your level of understanding as reflected by what you have written in the space provided.
- Please be neat and box your final answer, we cannot grade what we cannot decipher.

Name

Problem 1

A die is tossed. Find the probabilities of the events

1. $A = \{\text{odd number shows up}\}$
2. $B = \{\text{number larger than 3 shows up}\}$
3. $A \cap B$
4. $A \cup B$

Solution

The die is fair. Hence, $P[\{1\}] = P[\{2\}] = P[\{3\}] = P[\{4\}] = P[\{5\}] = P[\{6\}] = \frac{1}{6}$.

1. $P[A] = P[\{1,3,5\}] = \frac{3}{6} = \frac{1}{2}$.
2. $P[B] = P[\{4,5,6\}] = \frac{3}{6} = \frac{1}{2}$.
3. $P[A \cap B] = P[\{5\}] = \frac{1}{6}$.
4. $P[A \cup B] = P[\{1,3,4,5,6\}] = \frac{5}{6}$.

Problem 2

Computer programs are classified by the length of the source code and by the execution time. Programs with more than 150 lines in the source code are big (B). Programs with ≤ 150 lines are little (L). Fast programs (F) run in less than 0.1 seconds. Slow programs (W) require at least 0.1 seconds. Monitor a program executed by a computer. Observe the length of the source code and the run time. The probability model for this experiment contains the following information: $P[LF] = 0.5$, $P[BF] = 0.2$, $P[BW] = 0.2$. What is the sample space of the experiment? Calculate the following probabilities:

1. $P[W]$
2. $P[B]$
3. $P[W \cup B]$

Solution

We have

	L	B
W		0.2
F	0.5	0.2

$$P[LW] = 1 - \{P[BW] + P[LF] + P[BF]\} = 1 - 0.9 = 0.1.$$

$$P[W] = P[LW] + P[BW] = 0.1 + 0.2 = 0.3.$$

$$P[B] = P[BW] + P[BF] = 0.2 + 0.2 = 0.4.$$

$$P[W \cup B] = P[W] + P[B] - P[WB] = 0.3 + 0.4 - 0.2 = 0.5.$$

Problem 3

A short-circuit tester has a red light to indicate that there is a short circuit and a green light to indicate that there is no short-circuit. Consider an experiment consisting of a sequence of **three tests**. In each test the observation is the color of the light that is on at the end of a test. An outcome of the experiment is a sequence of red (r) and green (g) lights. Each outcome (a sequence of three lights, each either red or green) is **equally likely**. We denote each outcome by a three-letter word such as *rgr*, the outcome that the first and third lights were red but the second light was green. We denote the event that light n was red or green by R_n or G_n.

1. What is the sample space of the experiment?
2. Write the event R_2 that the second light is red
3. Write the event G_2 that the second light is green
4. Compute $P[R_2]$
5. Compute $P[G_2]$
6. Compute $P[R_2 \cap G_2]$

Are the events R_2 and G_2 independent? Are the events R_1 and R_2 independent?

Solution

1. $S = \{rrr, rrg, rgr, rgg, grr, grg, ggr, ggg\}$
2. $R_2 = \{rrr, rrg, grr, grg\}$.
3. $G_2 = \{rgr, rgg, ggr, ggg\}$.
4. $P[R_2] = \frac{4}{8} = \frac{1}{2}$.
5. $P[G_2] = \frac{4}{8} = \frac{1}{2}$.
6. We have $R_2 \cap G_2 = \emptyset$. Therefore, $P[R_2 \cap G_2] = 0$

Since $P[R_2 \cap G_2] \neq P[R_2]P[G_2]$, the events R_2 and G_2 are NOT independent.

$R_1 = \{rrr, rrg, rgr, rgg\}.P[R_1] = \frac{4}{8} = \frac{1}{2}$.

$P[R_1R_2] = P[\{rrr, rrg\}] = \frac{2}{8} = \frac{1}{4}$. On the other hand, $P[R_1]P[R_2] = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. Therefore, we have $P[R_1R_2] = P[R_1]P[R_2]$, and thus the events R_1 and R_2 are independent.

Problem 4

You have a six-sided die that you roll once. Let R_i denote the event that the roll is i. Let G_j denote the event that the roll is greater than j. Let E denote the event that the roll of the die is even-numbered.

1. What is $P[R_3|G_1]$, the conditional probability that 3 is rolled given that the roll is greater than 1?
2. What is the conditional probability that 6 is rolled given that the roll is greater than 3?
3. What is $P[G_3|E]$, the conditional probability that the roll is greater than 3 given that the roll is even?
4. Given that the roll is greater than 3, what is the conditional probability that the roll is even?

Solution

Let s_i denote the outcome that the roll is i. So, for $1 \leq i \leq 6$, $R_i = \{s_i\}$. Similarly, $G_j = s_{j+1}, \cdots, s_6$.

1. Since $G_1 = \{s2, s3, s4, s5, s6\}$ and all outcomes have probability $\frac{1}{6}$, $P[G_1] = \frac{5}{6}$. The event $R_3G_1 = \{s_3\}$ and $P[R_3G_1] = \frac{1}{6}$ so that
$$P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}$$
2. The conditional probability that 6 is rolled given that the roll is greater than 3 is
$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[s_6]}{P[\{s_4, s_5, s_6\}]} = \frac{1/6}{3/6} = \frac{1}{3}.$$
3. The event E that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability $3/6$. The joint probability of G_3 and E is $P[G_3E] = P[s_4, s_6] = 1/3$. The conditional probabilities of G_3 given E is
$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}.$$

4. The conditional probability that the roll is even given that its greater than 3 is
$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}.$$

Problem 5

Consider a binary code with 4 bits (o or 1) in each codeword. AN example of a code word is 0110.

1. How many different codewords are there?
2. How many codewords have exactly two zeros?
3. How many codewords begin with a zero?
4. In a constant-ratio binary code, each codeword has N bits. In every word, M of the N bits are 1 and the other $N - M$ bits are 0. How many different codewords are in the code with $N = 8$ and $M = 3$?

Solution

1. We can view choosing each bit in the code word as a subexperiment. Each subexperiment has two possible outcomes: 0 and 1. Thus by the fundamental principle of counting, there are $2 \times 2 \times 2 \times 2 = 24 = 16$ possible code words.

2. An experiment that can yield all possible code words with two zeroes is to choose which 2 bits (out of 4 bits) will be zero. The other two bits then must be ones. There are $\binom{4}{2} = 6$ ways to do this. Hence, there are six code words with exactly two zeroes. For this problem, it is also possible to simply enumerate the six code words:

$$1100, 1010, 1001, 0101, 0110, 0011.$$

3. When the first bit must be a zero, then the first subexperiment of choosing the first bit has only one outcome. For each of the next three bits, we have two choices. In this case, there are $1 \times 2 \times 2 \times 2 = 8$ ways of choosing a code word.

4. For the constant ratio code, we can specify a code word by choosing M of the bits to be ones. The other $N - M$ bits will be zeroes. The number of ways of choosing such a code word is $\binom{N}{M}$. For $N = 8$ and $M = 3$, there are $\binom{8}{3} = 56$ code words.

Problem 6

The random variable N has PMF

$$P_N(n) = \begin{cases} c/n, & n = 1, 2, 3; \\ 0, & \text{Otherwise.} \end{cases} \tag{1}$$

Find

1. the value of the constant c?
2. $P[N \geq 2]$
3. $P[N = 1]$
4. $P[N > 3]$

Solution

1. To find c, we recall that the PMF must sum to 1. That is,

   $$\sum_{n=1}^{2} P_N(n) = c(1 + \frac{1}{2} + \frac{1}{3}) = 1$$

   This implies $c = \frac{6}{11}$. Now that we have found c, the remaining parts are straightforward.

2. $P[N \geq 2] = P_N(2) + P_N(3) = \frac{c}{2} + \frac{c}{3} = \frac{5}{11}$
3. $P[N = 1] = P_N(1) = c = \frac{6}{11}$
4. $P[N > 3] = 0$, since the random variable N does not take any value that s strictly higher than 3.

Problem 7

Suppose we test 10 circuits and each circuit is rejected with probability $p = 1/4$ independent of the results of other tests. What is the probability that there exactly 3 circuits rejected?

Solution

Let X be the random variable denoting the number of rejected circuits.

$$\begin{aligned} P[X = 3] &= \binom{10}{3} p^3 (1-p)^7 \\ &= \binom{10}{3} (\frac{1}{4})^3 (\frac{3}{4})^7. \end{aligned}$$

Problem 8

The number of hits at a Web site in any time interval is a Poisson random variable. A particular site has on average $\lambda = 2$ hits per second.

1. What is the probability that there are no hits in an interval of 0.25 seconds?

2. What is the probability that there are no more than two hits in an interval of one second?

Recall that X is a Poisson(α) random variable if the PMF of X has the form

$$P_X(x) = \begin{cases} \frac{\alpha^x e^{-\alpha}}{x!}, & x = 0, 1, 2, \cdots; \\ 0, & \text{Otherwise.} \end{cases} \tag{2}$$

where the parameter $\alpha = \lambda T$: λ is the average rate per second and T a time interval.

Solution

1. In an interval of 0.25 seconds, the number of hits H is a Poisson random variable with $\alpha = \lambda T = 2 \times 0.25 = 0.5$ hits. The PMF of H is

   $$P_H(h) = \begin{cases} \frac{0.5^h e^{-0.5}}{h!}, & h = 0, 1, 2, \cdots; \\ 0, & \text{otherwise.} \end{cases}$$

   The probability of no hits is

   $$P[H = 0] = P_H(0) = (0.5)^0 e^{-0.5}/0! = e^{-0.5}.$$

2. In an interval of 1 second, $\alpha = \lambda T = 2 \times 1 = 2$ hits. Letting J denote the number of hits in sone second, the PMF of J is

   $$P_J(j) = \begin{cases} \frac{2^j e^{-2}}{j!}, & j = 0, 1, 2, \cdots; \\ 0, & \text{otherwise.} \end{cases}$$

   To find the probability of no more than two hits, we note that $\{J \leq 2\} = \{J = 0\} \cup \{J = 1\} \cup \{J = 2\}$ is the union of three mutually exclusive events. Therefore,

   $$\begin{aligned} P[J \leq 2] &= P[J = 0] + P[J = 1] + P[j = 2] \\ &= P_J(0) + P_J(1) + P_J(2) \\ &= e^{-2} + 2e^{-2}/1! + 2^2 e^{-2}/2! = 5e^{-2}. \end{aligned}$$

Extra credit problem worth 10 points

In an experiment with equiprobable outcomes, the event space is $S = \{1, 2, 3, 4\}$ and $P[s] = 1/4$ for all $s \in S$. Find three events in S that are pairwise independent but are not independent.

Solution

For a sample space $S = \{1, 2, 3, 4\}$ with equiprobable outcomes, consider the events $A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 1\}$. Each event A_i has probability 1/2. Moreover, each pair of events is independent since $P[A_1 A_2] = P[A_2 A_3] = P[A_3 A_1] = 1/4$. However, the three events A_1, A_2, A_3 are not independent since $P[A_1 A_2 A_3] = 0 \neq P[A_1]P[A_2]P[A_3]$.